The linear approximation of the λ-calculus: A new presentation of an old thing

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APPROXIMATING THE λ-CALCULUS?

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Here, a "syntactic" motivation:
to approximate the total dynamics ("information flow") of M
using pieces of finite dynamics ("finite information flows")

"Syntactic" approximation theorem (Wadsworth'78, Hyland'76, Barendregt):

$$\mathrm{BT}(M) = \lim \left\{ \begin{array}{c} \text{finite pieces of information} \\ \text{generated by } M \end{array} \right\}$$

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$= \bigsqcup \Big\{$  $\beta\perp$-normal $\lambda\perp$-term $\Big|$ $M \longrightarrow^{*}_{\beta}$  $\Big\}$.

"Commutation" theorem (Ehrhard-Regnier'06):

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The linear approximation

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... where $\mathcal{T} : \Lambda_\perp \to ?$ is defined by

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\mathcal{T}(x) &:= x\\
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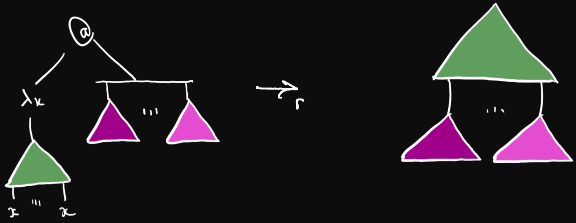
We need: multisets as arguments, sums of terms.

Resource terms:

$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad (s)\,[t_1, \ldots, t_n].$$

Resource reduction, featuring a multilinear substitution:



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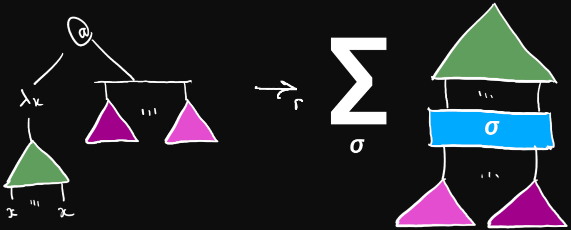

THE RESOURCE λ-CALCULUS

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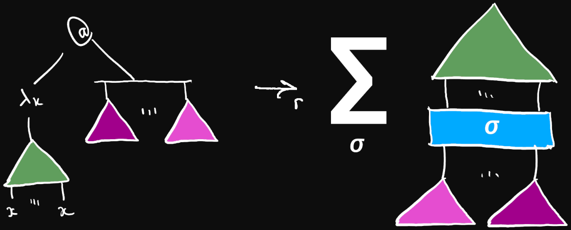
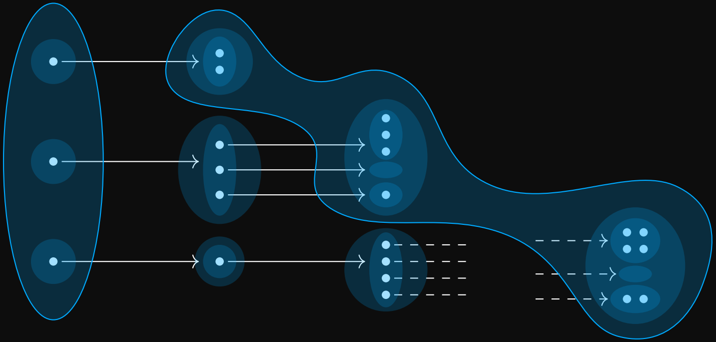


Excellent properties (confluence, normalisation)!

Finally, $\mathbf{S} \twoheadrightarrow_{\mathrm{r}} \mathbf{T}$ denotes the pointwise reduction (through $\longrightarrow_{\mathrm{r}}^{*}$) of possibly infinite sums of resource terms.



nf($\mathbf{S}$) is the pointwise normal form of $\mathbf{S}$.

Approximating the dynamics of the β-reduction

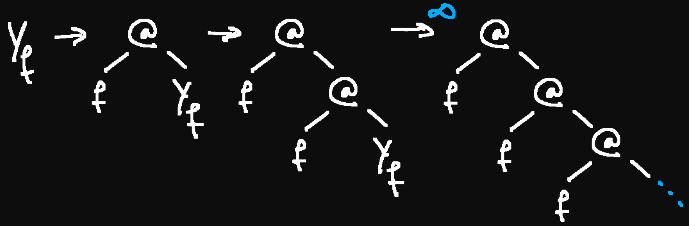
There's still something missing

Theorem (Vaux'17):
If $M \longrightarrow^*_{\beta\perp} N$ then $\mathcal{T}(M) \longrightarrow\!\!\!\!\rightarrow_{\mathrm{r}} \mathcal{T}(N)$.

This is not enough: we can't talk about $\mathrm{BT}(M)$...

- We still don't know what $\mathcal{T}(\mathrm{BT}(M))$ is.
- $\mathrm{BT}(M)$ may be infinitely far from M.

We want possibly infinite terms and reductions of a certain shape:



001-infinitary λ-terms, definition 1:

$$\frac{x \in \mathcal{V}}{x \in \Lambda_\perp^{001}} \qquad \frac{x \in \mathcal{V} \qquad M \in \Lambda_\perp^{001}}{\lambda x.M \in \Lambda_\perp^{001}} \qquad \frac{}{\perp \in \Lambda_\perp^{001}}$$

$$\frac{M \in \Lambda_\perp^{001} \qquad \triangleright N \in \Lambda_\perp^{001}}{MN \in \Lambda_\perp^{001}} \qquad \frac{N \in \Lambda_\perp^{001}}{\triangleright N \in \Lambda_\perp^{001}}$$

and we quotient by α-equivalence.

001-infinitary λ-terms, definition 2:

$$\Lambda_\perp^{001} := \nu Y.\mu X.\mathcal{V} + (\mathcal{V} \times X) + (X \times Y) + \perp$$

in the category of nominal sets (see C.'24).

001-infinitary closure of $\longrightarrow_\beta$:

$$\frac{M \longrightarrow_\beta^* x}{M \longrightarrow_\beta^{001} x} \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \qquad P \longrightarrow_\beta^{001} P'}{M \longrightarrow_\beta^{001} \lambda x.P'}$$

$$\frac{M \longrightarrow_\beta^* (P)Q \qquad P \longrightarrow_\beta^{001} P' \qquad \triangleright\, Q \longrightarrow_\beta^{001} Q'}{M \longrightarrow_\beta^{001} (P')Q'}$$

$$\frac{Q \longrightarrow_\beta^{001} Q'}{\triangleright\, Q \longrightarrow_\beta^{001} Q'}$$

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$$\begin{array}{c} Q \longrightarrow_\beta^{001} Q' \\ \hline\hline \triangleright\, Q \longrightarrow_\beta^{001} Q' \end{array}$$

Theorem (Kennaway *et al.*'97):

$\longrightarrow_{\beta\perp}^{\infty}$ is confluent, and the unique normal form of any $M \in \Lambda_\perp^{001}$ through $\longrightarrow_{\beta\perp}^{001}$ is $\mathrm{BT}(M)$.

One approximation theorem to rule them all

$\mathcal{T} : \Lambda_{\perp}^{001} \to \mathbb{S}^{\Lambda_\mathrm{r}}$ is defined (almost) as on finite terms (!).

$\mathcal{T} : \Lambda^{001}_{\perp} \to \mathbb{S}^{\Lambda_r}$ is defined (almost) as on finite terms (!).

"Simulation" theorem (C.-V.A.'23, C.'24):
If $M \longrightarrow^{\infty}_{\beta\perp} N$ then $\mathcal{T}(M) \twoheadrightarrow_{r} \mathcal{T}(N)$.

All the previous ones are easy consequences:

$$\begin{array}{ccc}
\text{simulation theorem} & \Longrightarrow & \text{commutation theorem} \\
\Downarrow & & \Downarrow \\
\text{confluence of } \longrightarrow^{\infty}_{\beta\perp} & \Longrightarrow & \text{syntactic approx. theorem}
\end{array}$$

And there's more!

Corollary

M has a HNF through $\longrightarrow_\beta^*$ or $\longrightarrow_\beta^{001}$
iff the head reduction strategy terminates on M
iff $\mathrm{nf}(\mathcal{T}(M)) \neq 0$.

Corollary

The Genericity lemma.

Corollary

$\mathrm{BT} : \Lambda_\perp^{001} \to \Lambda_\perp^{001}$ is Scott-continuous.

Let's get lazy

The lazy setting:

head normal forms	$\to$	weak head normal forms
Böhm trees	$\to$	Lévy-Longo trees
$\Lambda_{\perp}^{001}$	$\to$	$\Lambda_{\perp}^{101}$
$\longrightarrow_{\beta\perp}^{001}$	$\to$	$\longrightarrow_{\beta\perp}^{101}$

Example: $\mathsf{Y}_{\lambda y.\lambda x.y} \longrightarrow_{\beta}^{*} \lambda x.\mathsf{Y}_{\lambda y.\lambda x.y}$ is such that:

$$\mathrm{BT}(\mathsf{Y}_{\lambda y.\lambda x.y}) = \perp \qquad \mathrm{LLT}(\mathsf{Y}_{\lambda y.\lambda x.y}) = \mathsf{O} = \lambda x_0.\lambda x_1.\lambda x_2.\ldots$$

The **lazy resource λ-calculus**:

$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad \mathbb{0} \quad | \quad (s)\,[t_1, \ldots, t_n],$$

with $\quad (\mathbb{0})\,\bar{t} \longrightarrow_{\mathrm{r}} 0 \quad$ and $\quad \ell\mathcal{T}(\lambda x.M) := \lambda x.\ell\mathcal{T}(M) + \mathbb{0}.$

Theorem (simulation)

If $M \longrightarrow^{101}_{\beta\bot} N$ then $\ell\mathcal{T}(M) \twoheadrightarrow_{\ell\mathrm{r}} \ell\mathcal{T}(N)$.

Corollary (commutation)

$\mathrm{nf}(\ell\mathcal{T}(M)) = \ell\mathcal{T}(\mathrm{LLT}(M))$.

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Theorem (Severi-de Vries'05)

Only BT and LLT are Scott-continuous.

Conclusion

Linear approximation in a

- canonical,
- general,

presentation.

Slogan: treat infinitary stuff in an infinitary way!

What about...

- handling η-conversion?
- richer settings?
- cut-elimination in non-wellfounded proofs?

Thanks for your attention!